Due July 15, 11:59pm on Gradescope.

The following are warm-up exercises and are *not* to be turned in. You may treat these as extra practice problems.

5.1.13, 5.1.25, 5.1.37, 5.2.12, 5.3.7, 6.1.23, 6.1.35, 6.1.46, 6.2.18, 6.2.25, 6.2.42, 6.2.47.

Turn in the following exercises. Remember to carefully justify every statement that you write, and to follow the style of proper mathematical writing. You may cite any result proved in the textbook or lecture, unless otherwise mentioned. Each problem is worth 10 points with parts weighted equally, unless otherwise mentioned.

1. 5.1.26. You should do this with induction, even though there is a shorter non-inductive proof.

2. Suppose S is a nonempty subset of $\mathbf{Z}$ with the following properties:

   (1) S is closed under addition and subtraction: for any $s, s' \in S$, then $s + s', s - s' \in S$.

   (2) S absorbs multiplication: for any $s \in S$ and any integer a (not necessarily in S), we have $sa \in S$.

   Prove that there is an integer c such that S is exactly the subset of multiples of c.

   Remark: In more abstract terms, S is called an *ideal* of $\mathbf{Z}$, and what you have shown is that $\mathbf{Z}$ is a *principal ideal domain (PID).*

3. An *Egyptian fraction* is a fraction of the form $1/n$, where n is a positive integer. Prove that any fraction b/a, where $1 \leq b < a$, can be written as a finite sum of *distinct* Egyptian fractions (possibly a trivial sum involving only one fraction). [Hint: given b/a, write it as some $1/n + b'/a'$, where $1/n$ is the largest Egyptian fraction less than or equal to b/a, and b'/a' is the difference $b/a - 1/n$. Then do the same to b'/a', etc. Use strong induction to prove that this iterative process always terminates.]

4. 5.3.14.

5. 6.1.48 (3 points for (a), 3 points for (b), 4 points for (c)).

6. Let $n = 15015 = 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$. Find the number of integers $0 \leq x \leq 15014$ such that $x^2 \equiv 1 \bmod n$.

7. Here are some geometric applications of the pigeonhole principle:

   (a) Suppose we select 2024 points inside the (solid) unit cube $[0,1] \times [0,1] \times [0,1] \subseteq \mathbf{R}^3$. Show that there is a sphere of radius 1/11 that contains at least 3 of those points.

   (b) Suppose we select 7 real numbers $x_1, \ldots, x_7$, all within the interval $[-1,1]$. Show that there are distinct i and j in $\{1, 2, \ldots, 7\}$ such that

   $$x_i x_j + \sqrt{(1-x_i^2)(1-x_j^2)} \geq \frac{\sqrt{3}}{2}.$$

   [Hint: look at the 7 points $(x_i, \sqrt{1-x_i^2})$ on the upper hemisphere of the unit circle $x^2 + y^2 = 1$.]

8. **(Dirichlet Approximation Theorem)** Suppose r is a real number and $n \in \mathbf{N}$. Prove that there exist integers p, q such that $1 \leq q \leq n$ ans $|qr - p| < 1/n$. Deduce that there is a rational number p/q such that $1 \leq q \leq n$ and $|r - p/q| < 1/q^2$. [Hint: divide $[0,1)$ into n sub-intervals of equal length, and think about the fractional part of multiples of r, where the fractional part $\{x\}$ of a real number x is defined as $\{x\} \coloneqq x - \lfloor x \rfloor$. Equivalently and perhaps usefully, $\{x\}$ is the difference $x - a$ for the unique integer a such that $x - a \in [0,1)$.]

9. **(Bonus problem, 10 points)** Here is an application of the Dirichlet Approximation Theorem:

   (a) **(5 points)** Suppose r is an irrational number. Show that for any $x \in [0,1)$ and $n \in \mathbf{N}$, there is a positive integer q_n such that the fractional part $\{q_n r\}$ (see the hint for Problem 8) is within $1/n$ of x.

   (b) **(5 points)** Let k be a positive integer, and $a_1 a_2 \ldots a_k$ be an arbitrary string of digits of length k (so each of the a_i's is an integer between 0 and 9, and $a_1 \neq 0$), where not all the digits a_i are 9. Show that there exists a nonnegative integer power of 2024 whose first k digits are exactly $a_1 a_2 \ldots a_k$. [Hint: Prove that $\log_{10}(2024)$ is irrational, and try to apply part (a).]